On the Accuracy of Whitham's Method

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The steady flow of an ideal gas past a conical body is studied by the method of matched asymptotic expansions and by Whitham's method in order to assess the accuracy of the latter. It is found that while Whitham's method does not yield a correct asymptotic representation of the perturbation field to second order in regions where the flow ahead of the Mach cone of the apex is disturbed, it does correctly predict the changes of the second-order perturbation quantities across a shock (the first-order shock strength). The results of the analysis are illustrated by a special case of a flat, rectangular plate at incidence.

I. Introduction

THE method of Whitham[1,2] for computation of shock waves in supersonic flow has been employed extensively in recent years, especially in the study of sonic boom propagation.[3] While its simplicity of application is advantageous, analytical justification of the method often rests on a comparison of its predictions with results obtained by alternative analyses. It is recognized that the technique is inadequate to describe shock propagation involving focusing or diffraction-like effects,[3] but its accuracy in certain other circumstances is not fully established.

The present paper is devoted to an assessment of the accuracy of the Whitham theory in certain cases in which the flow ahead of the shock wave may be disturbed. For simplicity of presentation, only conical flow is considered, although extension of the results to describe the more general situation is, in principle, not difficult. The technique applied here is the method of matched asymptotic expansions,[4] which provides a rational perturbation scheme predicting the relative error at any stage of the perturbation process. When the results of the Whitham theory are expanded in variables corresponding to those used in the method of matched expansions, it can be determined by comparison whether Whitham's method yields a uniform approximation to the flowfield near the shock. In particular, it is found that in the vicinity of the Mach cone from the apex of a conical body the deviation of the field from linearized theory is correctly approximated only if the flow ahead of the surface of discontinuity is undisturbed. The Whitham theory does, however, predict the correct shock strength even when the flow ahead is disturbed.

II. Formulation

The problem to be considered is the steady flow of an ideal gas past a stationary, thin, conical body. The speed, pressure, and density of the approaching stream are, respectively, denoted by V_0, p_0, and ρ_0. The freestream Mach number is denoted by M, and a "conical" variable, r, is defined as $r = \beta\bar{r}/z$, where $(\bar{r}, \theta, z)$ form a cylindrical coordinate system, and $\beta = (M^2-1)^{1/2}$. Then the surface of a thin conical body with apex at the origin is defined by equations of the form $r\sin\theta = \varepsilon G(r\cos\theta)$ (in general, two such equations are required to define the upper and lower surfaces of the body). The small parameter ε introduced is a measure of the perturbations which the body produces in the undisturbed stream.

The equations governing the flow, which express the conservation of mass, linear momentum and energy, are well known and may be written in the form

$$\nabla\cdot(\bar{\rho}\bar{\mathbf{v}}) = 0 \quad \bar{\rho}\bar{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}+\nabla\bar{p} = 0$$
$$\bar{\mathbf{v}}\cdot\nabla[\bar{\mathbf{v}}\cdot\bar{\mathbf{v}}/2+(\gamma/(\gamma-1))\bar{p}/\bar{\rho}] = 0 \tag{1}$$

where $\bar{\mathbf{v}}$, $\bar{p}$, and $\bar{\rho}$ are, respectively, the velocity vector, pressure, and density. Further, a boundary condition which requires the velocity at the body surface to be tangent to that surface, and an initial condition which requires that the disturbances vanish ahead of the wave surface generated by the body, must be imposed. Dimensionless perturbation quantities are introduced as follows

$$u = \bar{v}_r/V_0 \quad v = \bar{v}_\theta/V_0 \quad w = \bar{v}_z/V_0 - 1 \quad p = (\bar{p}-p_0)/\gamma p_0$$
$$\rho = (\bar{\rho}-\rho_0)/\rho_0 \tag{2}$$

From the definition of ε, all these perturbation quantities tend to zero as ε approaches zero. Since only conical bodies are considered, "conical flow" solutions are sought: solutions where (u, v, w, p, ρ) depend on r and θ, but not on z. The equations of motion (1) in terms of the conical variables (r, θ) are listed in Ref. 5, where the formulation is discussed in more detail.

III. The Linearized Solution

Since the governing equations (1) are nonlinear, approximate solutions are sought by linearizing the equations on the hypothesis that the perturbations, Eq. (2), are small. The linearized equations are obtained immediately from Eq. (1) by dropping all terms containing products of the perturbations. Alternatively these equations can be obtained more formally by assuming that the exact solution to the problem has an "outer" asymptotic expansion of the form

$$u(r,\theta;\varepsilon) = \varepsilon u^{(1)}(r,\theta)+o(\varepsilon), \quad v(r,\theta;\varepsilon) = \varepsilon v^{(1)}(r,\theta)+o(\varepsilon), \ldots \text{ etc.} \tag{3}$$

The results of this linearized theory are discussed in Refs. 5 and 6 where it is shown that the solution may be expressed in terms of a "conical potential," $f(r, \theta)$, as

$$u^{(1)} = \beta\,\partial f/\partial r, \quad v^{(1)} = (\beta/r)\,\partial f/\partial\theta, \quad w^{(1)} = f - r\,\partial f/\partial r, \quad \text{and}$$
$$p^{(1)} = \rho^{(1)} = -M^2 w^{(1)} \tag{4}$$

where f satisfies $(1-r^2)f_{rr}+(1/r)f_r+(1/r^2)f_{\theta\theta} = 0$. Particular interest here is directed toward the conical surface $r = 1$, which is a characteristic of discontinuity in the linearized theory. In the vicinity of this characteristic, f is represented by expansions of the form[5,6]:

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$$f(r, \theta) = \begin{cases} D(\theta) - C(\theta)(1-r) + \frac{2}{3}B(\theta)(r-1)^{3/2} + \cdots \text{ for } r > 1 \\ D(\theta) - C(\theta)(1-r) - \frac{2}{3}A(\theta)(1-r)^{3/2} + \cdots \text{ for } r < 1 \end{cases} \quad (5)$$

The coefficients $A(\theta)$, $B(\theta)$, etc., are in general analytic functions, except at certain singular points. However, as noted in Ref. 6, the linearized theory does not provide a valid approximation to the flow near the wave front, $r = 1$. This difficulty can be traced to the square-root singularity in f_{rr}, and manifests itself physically in the prediction of a continuous field where shock discontinuities are known to exist.

IV. The Whitham Correction

Whitham,[1,2] proposed a simple, physically plausible, procedure for correcting the linearized solution so that it would provide a correct approximation to the flow near the wavefronts as well as away from them. There are several distinct steps in the Whitham theory. The first of these requires that the linearized solution be expressed in new independent variables: 1) a relevant linearized characteristic variable, $\bar{\xi}$, such that $\bar{\xi} = \text{const}$ is a characteristic of discontinuity in the linearized theory 2) a ray parameter which is constant on curves perpendicular to the linearized wavefronts, and 3) a variable measuring distance along a ray. For the problem at hand, these variables may be chosen as

$$1)\ \bar{\xi} = z - \beta\bar{r} \quad 2)\ \theta \quad 3)\ \bar{r} \quad (6)$$

and, expressed in these variables, the linearized solution takes the form

$$w = \varepsilon w^{(1)}(\bar{\xi}, \theta, \bar{r}) = \varepsilon\begin{cases} [D(\theta) - C(\theta)] - B(\theta)(-\bar{\xi}/\beta\bar{r})^{1/2} + \cdots \quad \text{for } \bar{\xi} < 0 \\ [D(\theta) - C(\theta)] - A(\theta)(\bar{\xi}/\beta\bar{r})^{1/2} + \cdots \quad \text{for } \bar{\xi} > 0 \end{cases} \quad (7)$$

with similar expressions for the remaining perturbation quantities.

Next, Whitham introduces the hypothesis that the linearized solution (7) will furnish a correct first approximation near the wavefront ($\bar{\xi} = 0$), as well as away from it, if $\bar{\xi}$ is replaced in the linearized solution by a variable ξ which is a better approximation to the nonlinear characteristics. Assuming that the characteristics $\xi = \text{const}$ lie close to their linearized position $\bar{\xi} = \text{const}$, ξ_θ is negligible compared to $\xi_{\bar{r}}$ and ξ_z, and a first improvement to the characteristics satisfies the same equation as given by Whitham[1] for the axisymmetric case, namely

$$(\partial z/\partial \bar{r})_{\xi,\theta} = \beta + \beta^2 K w - M^2(u + \beta w), \text{ where } K = \tfrac{1}{2}(\gamma+1)M^4\beta^{-3} \quad (8)$$

To furnish an initial condition for Eq. (8). Whitham requires that $\xi = \bar{\xi}$ when $\bar{r} = 0$. Thus, Whitham's hypothesis, together with Eqs. (7) and (8) yield

$$w = \varepsilon w^{(1)}(\xi, \theta, \bar{r}) = \varepsilon\begin{cases} D(\theta) - C(\theta) - B(\theta)(-\xi/\beta\bar{r})^{1/2} + \cdots \quad \text{for } \xi < 0 \\ D(\theta) - C(\theta) - A(\theta)(\xi/\beta\bar{r})^{1/2} + \cdots \quad \text{for } \xi > 0 \end{cases} \quad (9)$$

and so on, with

$$z - \beta\bar{r} = \xi + \varepsilon\left\{(\beta^2 K - M^2\beta)\int_0^{\bar{r}} w^{(1)}(\xi, \theta, t)\,dt - M^2\int_0^{\bar{r}} u^{(1)}(\xi, \theta, t)\,dt\right\} \quad (10)$$

Near the wavefront, as $(\xi/\beta\bar{r}) \to 0$, Eq. (10) reduces after some algebra to

$$z - \beta\bar{r} = \xi + \varepsilon\beta E(\theta)\bar{r} - \varepsilon 2\beta^{3/2} K F(\xi)(\bar{r})^{1/2} + \cdots \quad (11)$$

where

$$E(\theta) = -\beta K C(\theta) + (M^2/2\beta^2)[2 + (\gamma-1)M^2]D(\theta), \quad \text{and}$$

$$F(\xi) = \begin{cases} B(\theta)(-\xi)^{1/2} & \text{for } \xi < 0 \\ A(\theta)(\xi)^{1/2} & \text{for } \xi > 0 \end{cases}$$

Unless $A < 0$ and $B > 0$ the "corrected characteristics" [Eq. (11)] overlap, the approximation is multivalued, and the last step in Whitham's procedure is the construction of a shock based on the approximation that at each point the shock slope is the mean of the slopes of the characteristics ahead of and behind the shock. By methods discussed in Ref. 1, the shock $z = z_s(\bar{r}, \theta)$ is determined implicitly by

$$A\xi_2^{3/2} + B(-\xi_1)^{3/2} = \tfrac{3}{4}(\xi_2 - \xi_1)[A\xi_2^{1/2} + B(-\xi_1)^{1/2}]$$
$$z - \beta\bar{r} = \xi_1 + \varepsilon\beta E\bar{r} - \varepsilon 2\beta^{3/2} K B(-\xi_1\bar{r})^{1/2} \quad (12)$$
$$z - \beta\bar{r} = \xi_2 + \varepsilon\beta E\bar{r} - \varepsilon 2\beta^{3/2} K A(\xi_2\bar{r})^{1/2}$$

where $\xi_1 < 0$ and $\xi_2 > 0$. The conical nature of the Whitham approximation in this case can be made evident by introducing the parameter $\alpha = (\xi/\beta\bar{r})$. Then, for example, the pressure is given by

$$p = -\varepsilon M^2 w^{(1)}(\alpha, \theta) = -\varepsilon M^2\begin{cases} (D-C) - B(-\alpha)^{1/2} + \cdots (\alpha < 0) \\ (D-C) - A(\alpha)^{1/2} + \cdots (\alpha > 0) \end{cases} \quad (13)$$

where, from Eq. (11)

$$(1/r) - 1 = \alpha + \varepsilon E - \varepsilon 2\beta K F(\alpha) + \cdots \quad (14)$$

The shock defined by Eq. (12) may be determined from

$$A\alpha_2^{3/2} + B(-\alpha_1)^{3/2} = \tfrac{3}{2}\varepsilon\beta K\{A^2\alpha_2 + B^2\alpha_1\} \quad (15)$$

with $\alpha_1 < 0$ and $\alpha_2 > 0$ each satisfying Eq. (14). α may be expressed explicitly in terms of r and θ by solving Eq. (14), a quadratic equation on $(\pm\alpha)^{1/2}$; these in conjunction with Eq. (15) give the shock location, and the strength is determined from Eq. (13). If the values of A and B are not such as to produce a shock, there is a characteristic of discontinuity on $\alpha = 0$. In any case the curve of discontinuity (shock or characteristic) is represented by $r = 1 - \varepsilon E(\theta) + o(\varepsilon)$.

To examine the field in the neighborhood of this curve of discontinuity and to compare with the results to be derived in Sec. 5, an "inner expansion" is extracted from the Whitham approximation by defining the inner variables

$$X = \varepsilon^{-2}\{(1-r) - \varepsilon E(\theta)\} \quad \text{and} \quad \theta \quad (16)$$

and expanding the Whitham approximation for $\varepsilon \to 0$, holding X and θ fixed. To two terms this algebraic expansion yields

$$(u, w, p, \rho) = \varepsilon(U^{(1)}, W^{(1)}, P^{(1)}, R^{(1)}) + \varepsilon^2(U^{(2)}, W^{(2)}, P^{(2)}, R^{(2)}) + \cdots$$
$$\text{and} \quad v = \varepsilon V^{(1)} + \varepsilon^3 V^{(2)} + \cdots \quad (17)$$

where

$$U^{(1)} = \beta C, \qquad V^{(1)} = \beta\, dD/d\theta,$$
$$W^{(1)} = -M^{-2}P^{(1)} = -M^{-2}R^{(1)} = D - C$$
$$U^{(2)} = \beta\begin{Bmatrix} Bt_1^{1/2} \\ At_2^{1/2} \end{Bmatrix} \quad V^{(2)} = -\beta\begin{Bmatrix} -t_1 \\ t_2 \end{Bmatrix}(d/d\theta)(C-D) \quad \text{for} \begin{Bmatrix} X < X_d \\ X > X_d \end{Bmatrix} \quad (18)$$
$$w^{(2)} = -M^{-2}P^{(2)} = -M^{-2}R^{(2)} = -\beta^{-1}U^{(2)}$$

and

$$t_1 = \{[(\beta K B)^2 - (X + E^2)]^{1/2} - \beta K B\}^2$$
$$t_2 = \{[(\beta K A)^2 + (X + E^2)]^{1/2} + \beta K A\}^2 \quad (19)$$

$X_d(\theta)$ is the curve of discontinuity: a characteristic if $A < 0$ and $B > 0$ given by $X_d = -E^2(\theta)$, or a shock given implicitly by

$$At_2^{3/2} + Bt_1^{3/2} = \tfrac{3}{2}\beta K\{A^2 t_2 - B^2 t_1\} \quad (20)$$

together with Eq. (18). The shock strength is computed from Eqs. (17) and (18) as

$$p_2 - p_1 = (\bar{p}_2 - \bar{p}_1)/\gamma p_0 = \varepsilon^2 M^2\{Bt_1^{1/2} - At_2^{1/2}\} + \cdots \quad (21)$$

where $t_1(\theta)$, $t_2(\theta)$ are determined by Eqs. (19) and (20). Thus, the shock strength, a quantity of primary interest, is seen to be $0(\varepsilon^2)$, and the shock is seen to lie "in" the inner region. In regions where the linearized solution predicts undisturbed flow outside the Mach cone of the apex, the upper series in Eq. (5) vanishes identically. The results above may be specialized to this case simply by setting $B = C = D = 0$ (which implies $E = 0$), with one modification: since in this case $v^{(1)} = -\frac{2}{3}\beta(dA/d\theta)(1-r)^{3/2} + \ldots$, the inner expansion for v has the form

$$v = -\varepsilon^4 \tfrac{2}{3}\beta(dA/d\theta)t_2^{3/2} + \cdots \quad (X > X_d) \quad (21a)$$

An examination of the inner results, Eqs. (17–21a), shows a discrepancy. Brief reflection indicates that these results cannot be correct to second order. For example, in the case of the flat rectangular plate at an angle of incidence $\tan^{-1}\varepsilon$, the field ahead of the shock/characteristic is known exactly,[7] and has the form

$$v = \varepsilon V^{(1)} + \varepsilon^2 V^{(2)} + \cdots, \quad p = \varepsilon P^{(1)} + \varepsilon^2 P^{(2)} + \cdots \tag{22}$$

and so on, where the $V^{(i)}$ and $P^{(i)}$ are nonzero in the region of two-dimensional flow outside the influence range of the apex. But for this case B vanishes [along with all succeeding coefficients in the upper series, Eq. (5)], and therefore the Whitham approximation indicates that, in the plane flow region

$$v = \varepsilon V^{(1)} + o(\varepsilon^3), \quad p = \varepsilon P^{(1)} + o(\varepsilon^2) \quad \text{etc.} \tag{23}$$

On the other hand it will be shown that the shock strength obtained from Eqs. (19–21) agrees exactly with that found by Lighthill[6] using his method of coordinate straining. This raises the question: if Whitham's procedure does not give correct asymptotic representations to second order for the flow variables ahead of and behind the shock, does it predict correctly the *changes* in these quantities across the shock (the shock strength), which are of second order, $[0(\varepsilon^2)]$? This problem can be illuminated by a study of the flow near the wavefronts via the method of matched asymptotic expansions, which is explained in Ref. 4.

V. The Inner Expansion

The method of matched asymptotic expansions seeks to correct the deficiency of the linearized solution (5), which is the first term of an outer asymptotic expansion, by providing supplementary inner expansions to represent the solution in regions where the outer expansion is not valid: in this case, near $r = 1$. Such inner expansions are then assumed to be related to the contiguous outer expansions by the asymptotic matching principle stated in Ref. 4. The first step in this procedure requires that new independent variables be defined which are $0(1)$ in the region of nonuniformity. An appropriate definition for such variables is

$$\bar{X} = \varepsilon^{-1}(1-r) \quad \text{and} \quad \theta \text{ itself} \tag{24}$$

Examination of the linearized solution with regard to matching considerations indicates that the exact solution has an expansion of the form

$$(u, w, p, \rho) = \varepsilon(\bar{U}^{(1)}, \bar{W}^{(1)}, \bar{P}^{(1)}, \bar{R}^{(1)}) + \varepsilon^{3/2}(\bar{U}^{(2)}, \bar{W}^{(2)}, \bar{P}^{(2)}, \bar{R}^{(2)}) + \cdots \quad \text{and}$$
$$v = \varepsilon \bar{V}^{(1)} + \varepsilon^2 \bar{V}^{(2)} + \cdots \tag{25}$$

where $\bar{U}^{(1)}$, $\bar{V}^{(1)}$, etc., are functions of $\bar{X}$ and θ. Of course, the asymptotic sequence of gauge functions in Eq. (25) could be left unspecified. In that case the form Eq. (25) would result upon application of matching conditions with the outer expansion.

Equations for the determination of $\bar{U}^{(1)}$, $\bar{V}^{(2)}$, etc., can be obtained by inserting the expansions (25) into the conical form of the field equations (1), and equating the coefficients of like powers of ε. This is a tedious, but purely mechanical, procedure, and it is omitted here. However, one difficulty which arises at this stage must be noted. Direct expansion of the conical field equations (1) written in terms of conical variables, leads to a redundant set of first-order equations from which $\bar{U}^{(1)}, \ldots, \bar{R}^{(1)}$ cannot be determined (this problem is discussed further in Ref. 5). This difficulty can be overcome as follows: the equation

$$c^2 \nabla \cdot \bar{\mathbf{v}} = \bar{\mathbf{v}} \cdot \nabla(\tfrac{1}{2}\bar{\mathbf{v}} \cdot \bar{\mathbf{v}}), \quad \text{where} \quad c^2 = \gamma \bar{p}/\bar{\rho} \tag{26}$$

can be derived from Eq. (1) by a manipulation not involving differentiation of any equations of (1). Equation (26) is then transformed into conical variables (r, θ), which yields an equation of the form

$$a_1(q)u_r + a_2(q)v_r + a_3(q)w_r + b_1(q)u_\theta + b_2(q)v_\theta + b_3(q)w_\theta + c(q) = 0 \tag{27}$$

where $q = (u, v, w, p, \rho)$. Next, using the momentum equation from (1), it can be shown that

$$w_r = -(r/\beta)u_r - (1/M^2 r)\{u - (r/\beta)(1+w)\}^{-1} \times \{M^2 v w_\theta + (r/\beta)(M^2 v u_\theta - M^2 v^2)\} \tag{28}$$

If w_r in Eq. (27) is replaced by its expression in Eq. (28) the following equation results

$$\begin{aligned}&[\beta^2 r(1+\gamma p) - M^2\beta r u(1+p)\{\beta u - r(1+w)\} + r^3(1+\gamma p) - \\ &M^2 r(1+\rho)(1+w)\{r(1+w) - \beta u\}]u_r + \beta^2(1+\gamma p)(u+v_\theta) - \\ &M^2\beta^2 v w_\theta - (\beta/M^2 r)\{u - (r/\beta)(1+w)\}^{-1} \times \\ &\{M^2 v w_\theta + (r/\beta)(M^2 v u_\theta - M^2 v^2)\}[M^2 r(1+p)\{r(1+w) - \beta u\} \times \\ &(1+w) - r^2(1+\gamma p)] + M^2\beta r v v_r(1+\rho)\{r(1+w) - \beta u\} - \\ &M^2\beta^2 \rho v w_\theta - M^2\beta v(1+\rho)(u u_\theta + v v_\theta + w w_\theta) = 0\end{aligned} \tag{29}$$

The last of Eq. (1) is replaced by Eq. (29). This new set of five field equations is equivalent to Eq. (1) (in conical form) and when the expansions, Eq. (25), are inserted in these equations, there result sets of five independent equations for the determination of $\bar{U}^{(1)}, \ldots, \bar{R}^{(1)}$, etc. Equation (29) has the advantage that it leads directly to simple inner equations on $\bar{U}^{(1)}$ and $\bar{U}^{(2)}$.

These equations can easily be integrated, and when the matching conditions of Ref. 4 are imposed between the expansion, Eq. (25), and the one-term outer expansion (linearized solution), most of the arbitrary functions of integration can be determined and one finds

$$\begin{aligned}&\bar{U}^{(1)} = \beta C, \quad \bar{V}^{(1)} = \beta\, dD/d\theta, \\ &\bar{W}^{(1)} = -M^{-2}\bar{P}^{(1)} = -M^{-2}\bar{R}^{(1)} = D - C \\ &\bar{U}^{(2)} = \beta\begin{cases}B(E-\bar{X})^{1/2}; & \bar{X} < E \\ A(\bar{X}-E)^{1/2}; & \bar{X} > E\end{cases}, \\ &\bar{V}^{(2)} = \bar{g}(\theta), \ \bar{W}^{(2)} = -M^{-2}\bar{P}^{(2)} \\ &\qquad = -M^{-2}\bar{R}^{(2)} \\ &\qquad = -\beta^{-1}\bar{U}^{(2)}\end{aligned} \tag{30}$$

where $E(\theta)$ is defined in Eq. (11).

The arbitrary function of integration $\bar{g}(\theta)$ cannot be determined by matching with the one-term outer expansion: higher order terms in the outer expansion are required. For reasons discussed in Ref. 6, the appearance of a square root singularity in the second terms of Eq. (25) indicates that this expansion is invalid sufficiently close to $\bar{X} = E(\theta)$, where shocks or characteristics of discontinuity are expected; the singularity will be compounded in higher order terms of the expansion. For this reason Eq. (25) is designated the "middle expansion," and an "inner expansion" is sought to describe the flow near $\bar{X} = E(\theta)$. To this end, an "inner variable" is defined

$$X = \varepsilon^{-1}\{\bar{X} - E(\theta)\} = \varepsilon^{-2}\{(1-r) - \varepsilon E(\theta)\} \tag{31}$$

so that the "size" of the inner region is $0(\varepsilon^2)$ about $1 - r = E(\theta)$.

An inner expansion, which will satisfy matching conditions with the expansion Eq. (25), is sought of the form

$$(u, w, p, \rho) = \varepsilon(U^{(1)}, W^{(1)}, P^{(1)}, R^{(1)}) + \varepsilon^2(U^{(2)}, W^{(2)}, P^{(2)}, R^{(2)}) + \cdots \tag{32}$$
$$v = \varepsilon V^{(1)} + \varepsilon^2 V^{(2)} + \varepsilon^{5/2} V^{(3)} + \varepsilon^3 V^{(4)} + \cdots$$

where the $U^{(i)}$, $V^{(i)}$, etc., are functions of X and θ only.

The first-order problem can be easily formulated and solved; matching conditions are imposed between the one-term inner and middle expansions yielding the solution

$$\begin{aligned}&U^{(1)} = \beta C, \qquad V^{(1)} = \beta\, dD/d\theta, \\ &W^{(1)} = -M^{-2}P^{(1)} = -M^{-2}R^{(1)} = D - C\end{aligned} \tag{33}$$

The second-order inner problem will be considered in more detail. The set of five second-order inner equations can be written in divergence form as

$$\frac{\partial}{\partial X}\begin{bmatrix} P^{(2)} - (M^2/\beta)U^{(2)} \\ R^{(2)} - (M^2/\beta)U^{(2)} \\ W^{(2)} + (1/\beta)U^{(2)} \\ V^{(2)} \\ (Q - KU^{(2)})U^{(2)} \end{bmatrix} + \frac{\partial}{\partial\theta}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -V^{(2)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3U^{(2)} - \psi_2(\theta) \end{bmatrix} = 0 \tag{34}$$

where

$$Q = 2\{X-\psi_1(\theta)+(M^2/2\beta^2)[2(\beta U^{(2)}-W^{(2)})+\gamma P^{(2)}-R^{(2)}]\}$$

$$\psi_1(\theta) = \tfrac{3}{2}E^2+(2\beta^2)^{-1}\gamma(2+M^2)EP^{(1)}+(2\beta^2)^{-1}\times M^2\{(W^{(1)}+R^{(1)})(W^{(1)}-\beta U^{(1)})+R^{(1)}W^{(1)}-\beta U^{(1)}(W^{(1)}+R^{(1)}-\beta U^{(1)}-2E)-E(6W^{(1)}-2\beta U^{(1)}+3R^{(1)})\}-\tfrac{1}{2}(dE/d\theta)[1+2(M^2/\beta)V^{(1)}]$$

$$\psi_2(\theta) = -(dW^{(1)}/d\theta)[\beta(dE/d\theta)-2M^2V^{(1)}]$$

The shock conditions satisfied by weak solutions of Eq. (34) follow immediately.[8] Assuming $|dX/d\theta| < \infty$ on the shock, these conditions are

$$\left[P^{(2)}-\frac{M^2}{\beta}U^{(2)}\right]=\left[R^{(2)}-\frac{M^2}{\beta}U^{(2)}\right]=\left[W^{(2)}+\frac{1}{\beta}U^{(2)}\right] = [V^{(2)}] = [(Q-KU^{(2)})U^{(2)}] = 0 \quad (35)$$

where the brackets denote the jumps in the quantities across the shock. The first four equations of (34) may be immediately integrated to yield

$$R^{(2)} = \frac{M^2}{\beta}U^{(2)}+g_1(\theta),\quad P^{(2)} = \frac{M^2}{\beta}U^{(2)}+g_2(\theta),$$

$$W^{(2)} = -\frac{1}{\beta}U^{(2)}+g_3(\theta),\quad V^{(2)} = g_4(\theta) \quad (36)$$

and the first four shock conditions of Eq. (35) require that all the $g_i(\theta)$ be continuous across any shocks. Using Eq. (36), the last equation of (34) may be written as

$$2\{X-\phi(\theta)+KU^{(2)}\}U_X{}^{(2)}-U^{(2)} = \psi(\theta) \quad (37)$$

where

$$\phi(\theta) = \psi_1(\theta)+(M^2/2\beta^2)(2g_3+g_1-\gamma g_2)$$

and

$$\psi(\theta) = \psi_2(\theta)+dg_4/d\theta.$$

If $U_X{}^{(2)} \neq 0$, the general solution of Eq. (37) is

$$U^{(2)}(X,\theta) = \frac{1}{2g_5(\theta)}\times\left[1\pm(1-2g_5(\theta)\{\psi(\theta)-K^{-1}[X-\phi(\theta)]\})^{1/2}\right]-\psi(\theta) \quad (38)$$

where $g_5(\theta)$ is an arbitrary function of integration. This function may be determined by imposing the matching condition that the two-term inner expansion of (the two-term middle expansion) be equal to the two-term middle expansion of (the two-term inner expansion). This yields

$$U^{(2)} = \begin{cases}\beta B\{[(\beta KB)^2-(X-\Gamma)]^{1/2}-\beta KB\}-\psi; & X < X_d(\theta)\\ \beta A\{[(\beta KA)^2+(X-\Gamma)]^{1/2}+\beta KA\}-\psi; & X > X_d(\theta)\end{cases} \quad (39)$$

where $\Gamma(\theta) = \phi+K\psi$, and $X_d(\theta)$ represents the shock or characteristic of discontinuity. Equations (36) and (39) give the *form* of the second-order inner solution; however, it does not appear possible to determine the arbitrary functions $g_1(\theta)$ through $g_4(\theta)$ without having available, at least in the region outside the range of influence of the apex, higher-order terms of the outer expansion. These are, in general, difficult to obtain, but they are readily available for a few simple situations such as the flat rectangular plate at incidence to be considered shortly. The characteristic is the curve on which derivatives of $U^{(2)}$, and of the other flow variables, are discontinuous; from Eq. (39) this curve is given by $X_d = \Gamma(\theta)$. If a shock exists, then the last of the shock conditions (35) must be satisfied on it. Auxiliary variables are defined as follows:

$$\bar{F}(X,\theta) = (\beta K)^{-2}[\Gamma(\theta)-X] \quad (40)$$

$$\bar{z}_1(X,\theta) = (B^2+\bar{F})^{1/2}-B,\quad \bar{z}_2(X,\theta) = (A^2-\bar{F})^{1/2}+A \quad (41)$$

Then, on the shock,

$$\bar{F} = \bar{F}[X_s(\theta),\theta] = F(\theta),\quad \bar{z}_1 = z_1(\theta),\quad \bar{z}_2 = z_2(\theta)$$

and

$$2Bz_1+z_1{}^2 = 2Az_2-z_2{}^2 \quad (42)$$

$U^{(2)}$ may be written as:

$$U^{(2)} = \begin{cases}\beta^2KB\bar{z}_1-\psi; & X < X_d\\ \beta^2KA\bar{z}_2-\psi; & X > X_d\end{cases} \quad (43)$$

In terms of these auxiliary variables, the last shock condition becomes

$$\{\beta^2KBz_1-\psi\}\{-3\beta^2K^2Bz_1-2\beta^2K^2z_1{}^2+K\psi\} = \{\beta^2KAz_2-\psi\}\{-3\beta^2K^2Az_2+2\beta^2K^2z_2{}^2+K\psi\}$$

which, using condition (42) reduces to

$$Az_2{}^3+Bz_1{}^3 = \tfrac{3}{2}\{A^2z_2{}^2-B^2z_1{}^2\} \quad (44)$$

Equations (42) and (44) determine z_1 and z_2 on the shock as functions of θ, and therefore determine $F(\theta)$. However, the shock location cannot be determined to second order from Eq. (40) because $\Gamma(\theta)$ contains the undetermined functions g_1, g_2, g_3. On the other hand the shock *strength* may be determined regardless of these undetermined functions as follows:

$$\frac{\bar{p}_2-\bar{p}_1}{\gamma p_0} = \left\{\varepsilon(D-C)+\varepsilon^2\left(\frac{M^2}{\beta}U_2{}^{(2)}+g_2\right)+\cdots\right\}-\left\{\varepsilon(D-C)+\varepsilon^2\left(\frac{M^2}{\beta}U_1{}^{(2)}+g_2\right)+\cdots\right\}$$

or

$$\frac{\bar{p}_2-\bar{p}_1}{p_0} = \varepsilon^2\frac{\gamma(\gamma+1)M^6}{2\beta^2}\{Az_2-Bz_1\}+\cdots \quad (45)$$

Thus Eqs. (42, 44, and 45) define the shock strength, independently of the second-order shock location, a situation analogous to that found by Lighthill in Ref. 6. The pair of Eqs. (42) and (44) can be shown to be equivalent to

$$2Bz_1+z_1{}^2 = 2Az_2-z_2{}^2 = \tfrac{1}{2}(Az_2+Bz_1) \quad (46)$$

From Eq. (41) and the second of Eqs. (46) the following equation on $F(\theta)$ results

$$A(A^2-F)^{1/2}+B(B^2+F)^{1/2}+A^2-B^2 = 2F \quad (47)$$

Equations (45) and (47), which define the shock strength as a function of θ are exactly the same as those derived and discussed in Ref. 6.

In regions where the linearized solution indicates that the flow outside the Mach cone of the apex is undisturbed, the upper series of Eq. (5) vanishes identically. Here the exact solution vanishes for $X < X_d$, implying that $g_1 = g_2 = g_3 = g_4 = E = \Gamma = 0$, and the leading terms in the expansion of u, w, p, and ρ, which are all $0(\varepsilon^2)$, may be obtained directly from Eqs. (36) and (39). (The middle expansion is superfluous in this case.) Equation (36) indicates only that $v = o(\varepsilon^2)$. This special case has been studied in detail in Ref. 5, where it is shown that, in fact,

$$v = -\varepsilon^4\tfrac{2}{3}\beta(dA/d\theta)\{[(BKA)^2+X]^{1/2}+\beta KA\}^3+\cdots \quad (48)$$

If a shock exists, its location and strength can be determined to $0(\varepsilon^2)$ from the formulas of this section by setting $B = 0$, which yields

$$X_d = -\tfrac{3}{4}(\beta KA)^2 \quad \text{and } (1)\bar{p}_2-\bar{p}_1(1)/p_0 = \varepsilon^2\tfrac{3}{4}\gamma(\gamma+1)M^6\beta^{-2}A^2+\cdots \quad (49)$$

VI. Discussion

In regions where the flow ahead of the Mach cone of the apex is undisturbed, comparison of the inner expansion obtained in the last section, with that extracted from the Whitham approximation show that the two do not agree to second order, although the first-order terms coincide. Thus, the second-order terms for v cannot agree unless $g_4(\theta)$ vanishes; similarly the second-order expressions for u cannot agree unless $\psi(\theta)$ vanishes and $\Gamma(\theta) = -E^2(\theta)$. That these conditions do not hold in general can be seen from the example of the flat rectangular plate discussed below. In these regions the first-order inner field is simply the limit of the linearized field as $r \rightarrow 1$: Whitham's theory does not predict deviations from these linearized values correctly. On the other hand, in regions where the flow outside the Mach cone of the apex is undisturbed, the $0(\varepsilon)$ terms vanish, and the Whitham results agree exactly with those obtained by the method of matched asymptotic expansions. Finally comparison of the expressions for the shock strength [Eqs. (19–21, 42, 44, and 45] show that the shock strengths deduced by both methods agree, whether or not the flow ahead of the shock is undisturbed. However, Whitham's method does not give the location of the shock correctly to $0(\varepsilon^2)$.

As previously noted, the arbitrary functions arising in the integration of the second-order inner problem ultimately must be determined by matching with the two-term outer expansion. For illustration, a simple example for which the matching may be done explicitly will be used: namely, a flat, rectangular plate at an angle of incidence $\tan^{-1}\varepsilon$ to the oncoming stream. In this case the exact solution is known[7] in the region of two-dimensional flow outside the range of influence of the apex (shock/characteristic of discontinuity). This solution is zero for $\pi/2 < \theta < \frac{3}{2}\pi$, and

$$\begin{aligned} u &= \varepsilon(\mp \sin\theta) + \varepsilon^2(-\beta^{-1}\sin\theta) + \cdots \\ v &= \varepsilon(\mp \cos\theta) + \varepsilon^2(-\beta^{-1}\cos\theta) + \cdots \\ w &= \varepsilon(\pm\beta^{-1}) + \varepsilon^2\{\beta^{-2}(1-\tfrac{1}{2}\beta K)\} + \cdots \\ p &= \varepsilon(\mp M^2\beta^{-1}) + \varepsilon^2\{-M^2\beta^{-2}(1-\tfrac{1}{2}\beta K)\} + \cdots \\ \rho &= \varepsilon(\mp M^2\beta^{-1}) + \varepsilon^2\{M^2(4\beta^2)^{-1}[\gamma(1-\beta^4)+(1+3\beta^4)]\} + \cdots \end{aligned} \tag{50}$$

The upper signs apply in the region $0 < \theta < \pi/2$, while the lower signs apply in $\frac{3}{2}\pi < \theta < 2\pi$. (The Prandtl-Meyer expansion fan above the plate is disregarded in this analysis.) As Eq. (50) holds up to the shock/characteristics, it is simultaneously the two-term outer, middle, and inner expansion. For this case $B = 0$, and from Eqs. (36, 39, and 50), it follows immediately that

$$\psi(\theta) = \beta^{-1}\sin\theta, \qquad g_4(\theta) = -\beta^{-1}\cos\theta \quad \text{etc.} \tag{51}$$

The curve $\Gamma(\theta)$ which gives the location of the shock/characteristic to $0(\varepsilon^2)$ can be computed from the expressions given previously, but this step is omitted here. The second-order inner perturbation pressure, π, is defined as $\pi = \varepsilon^{-2}(p - \varepsilon P^{(1)})$. Then the jump in π across the shock, a measure of the shock strength, is from Eq. (45)

$$\pi_2 - \pi_1 = \frac{1}{\varepsilon^2\gamma}\left(\frac{\bar{p}_2 - \bar{p}_1}{p_0}\right) = \frac{3}{2}K\beta M^2 A^2(\theta) + \cdots \tag{52}$$

(The subscripts 1 and 2 refer to quantities evaluated on the front and back side of the shock, respectively.)

Figure 1 displays these results graphically. For the flat plate, $A(\theta)$ is, from Ref. 6,

$$A(\theta) = -\tan\theta/\beta\pi\sin(\theta/2)$$

The values of the parameters chosen are $M = 2$, $\gamma = 1.4$, and $\pi_0 = \pi_1(\theta = 0) = -M^2\beta^{-2}(1-\beta K/2)$. In this simple case of plane boundaries, π_1 is a constant, whereas, of course, for curved boundaries it varies. The figure shows that Whitham's technique gives π_1 incorrectly as identically zero in this case, but that it does give $(\pi_2 - \pi_1)$ (the shock strength) correctly.

It should be remarked that Whitham's approximation may be obtained as a first term of an expansion of the solution in terms of characteristic parameters, studied by Lin[9] in two-dimensional problems. Viewed in this context, it becomes clear that second-order terms in this expansion can contribute to second-order terms in the inner expansion, and therefore it should not be expected that the Whitham approximation contains the *two*-term inner expansion. Thus it is remarkable that the Whitham approximation gives the shock strength correctly to $0(\varepsilon^2)$. The nonuniformities of the outer expansion near plane wavefronts, and near the points of tangency of these wavefronts and the Mach cone of the apex have not been considered here. These problems are further discussed in Ref. 5.

VII. Conclusions

A comparison of results obtained by Whitham's technique and an analysis by the method of matched asymptotic expansions, leads to the following conclusions in the case of flow near the Mach cone from the apex of a thin conical body:

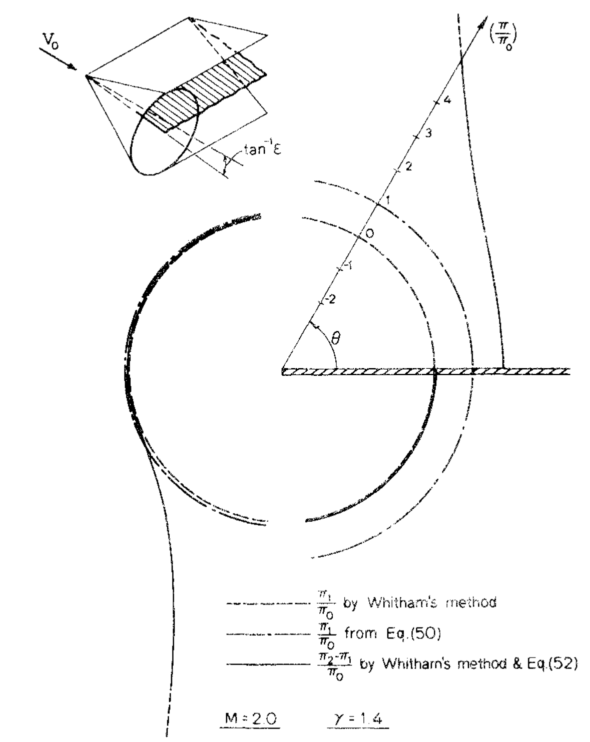

Fig. 1 Second-order pressure and shock strength near the Mach cone from the apex of a flat, rectangular plate at incidence.

1) The Whitham technique gives a correct first approximation for the flow variables everywhere.

2) The Whitham technique does not give a correct second approximation for the flow variables in regions where the flow outside the Mach cone is disturbed.

3) Nevertheless, the Whitham technique correctly gives the *discontinuity* in the second approximation to the flow variables across a shock in the regions (2). Thus Whitham's method gives a correct first approximation to the shock strength everywhere.

4) Whitham's method gives a correct first (but not second) approximation to the displacement of the shock/characteristic from its linearized position.

The terms first and second approximation are used here in the precise sense of asymptotic limits described in the fifth section.

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